On the Topology of Lorentzian manifolds

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Abstract

In this essay we review some of the alternative topologies that have been considered on Lorentzian manifolds in order to embed the causal structure of spacetime on a more fundamental level. We give a possible explanation of why none of these topologies are metrizable. We then make an attempt to generalize the formalism of Riemannian geometry to a "causal" topology by constructing "causal manifolds" and we explain why this fails. Then we turn our attention to the tangent spaces and discuss how we might still be able to define a tangent space such that straight lines are locally homeomorphic to geodesics. Finally we take a look at the possibility of endowing such a tangent space with a different vector space structure that encompasses the structure of relativistic addition of velocities. At the end of this essay we discuss possible implications of these mathematical considerations for the field of quantum gravity.

Introduction

Behind the world of general relativity that Einstein created in the early twentieth century, seems to lie a beautiful mathematical structure that was already available well before Einstein's discovery: that of Riemannian geometry. Riemannian geometry is the study of manifolds that are endowed with a Riemannian metric, which is a positive-definite inner product on the tangent spaces. It happens to be so that in Riemannian geometry the different induced notions of a *topology* coincide very nicely. On the one hand, in order to be a manifold, the space must locally have the same topology as $\mathbb{R}^n$. On the other hand, the Riemannian metric itself also induces a topology - it induces *distances* on the manifold, which induce a basis of topology via the collection of open balls with different radii. This topology is the same as the previous one. Then thirdly, the Riemannian metric also induces a unique diffeomorphism that preserves the lengths of some geodesics between the tangent space at a point and a neighborhood of the point in the manifold: the *exponential map*. The tangent space is $\mathbb{R}^n$ viewed as a vector space, and therefore has a unique Hausdorff n-dimensional topology as a topological vector space. This topology is again that of $\mathbb{R}^n$, and this is locally carried down to the manifold by the exponential map, so all three notions of a topology coincide on a Riemannian manifold.

Unfortunately, the setting of general relativity is in fact not Riemannian geometry at all. It is an in many aspects analogous area of mathematics called pseudo-Riemannian geometry, where we endow the tangent bundle with a bilinear structure that is symmetric and nondegenerate, but not positive-definite. For a long time this subject was not very popular with mathematicians [1, 2], which is not very surprising since, though many theorems from Riemannian geometry can be generalized to these spaces, the structure is just not as beautiful. For example, pseudo-metrics do not in general induce a topology, so there is no nice generated topology to coincide locally with the topology of $\mathbb{R}^n$. General relativity restricts itself to the subclass of Lorentzian manifolds, where the pseudo-metric is at every point conjugate to a metric with trace $n-2$, where n is the dimension of the spacetime.

There are a few reasons why we might consider giving such a manifold a different topology. Let us consider the simplest example of a Lorentzian manifold, namely Minkowski space. Minkowski space is usually endowed with the normal Euclidean topology of $\mathbb{R}^n$. But $\mathbb{R}^n$ is locally homogeneous, whereas Minkowski space has an intrinsic general direction of time, or a light cone, attached to each point. Another argument is the fact that the group of homeomorphisms of $\mathbb{R}^n$ contains all kinds of elements that exchange spacelike and timelike directions, which is not an action that is physically allowed. But we would like to give this homeomorphism group[1], and the diffeomorphism group in the differentiable case, a physical interpretation that could be useful in constructing a theory of quantum gravity [3]. This can only be done if the elements at least leave the causal structure of Minkowski space invariant, and therefore the topology would need to have an intrinsic incorporation of causality.

But changing the topology of a space is in some sense equivalent to changing what functions and curves are “nice”. In particular, if we change the topology on a Lorentzian manifold, then it will not be possible to define a “nice” coordinate system $x^\mu(p)$ at each point. This is because the space is not locally homeomorphic to $\mathbb{R}^n$ anymore, so there will be no functions that are continuous and continuously invertible from a subset of the space to a subset of $\mathbb{R}^n$. So if we introduce such a new topology on a Lorentzian manifold, the space will no longer be an actual manifold. However, we might try to define another coordinate system that does not locally map our space to $\mathbb{R}^n$, but to some other space, with which it is locally homeomorphic.

This would amount to trying to construct a different concept of a manifold: a *causal manifold*, a space that is locally, in coordinate neighborhoods, diffeomorphic to a new “Minkowski space” $\mathbb{M}^n$ with a different “causal” topology, such that we have smooth transition maps between the coordinate charts. These diffeomorphisms then form the coordinate system that maps points in the Lorentzian space to points in $\mathbb{M}^n$ with their “Minkowski coordinates”.
Such a causal manifold would have an intrinsic notion of causality, even before we endow it with a pseudo-Riemannian structure, if we incorporate this into the topology of $\mathbb{M}^n$. In fact, even without the coordinate functions, the topology of

[1]Mathematicians usually call this the automorphism group, as it consists of homeomorphisms from the space to itself.

the manifold would then encode causality, because it would locally be the same as the causal topology of $\mathbb{M}^n$.
But what should this topology on $\mathbb{M}^n$ be? Is it possible to create such a structure at all? In particular, can a topology be restrictive enough to incorporate causality, but lenient enough to allow the causal manifold to be curved? The answer is in fact that this probably can not be done.

Apart from a local coordinate system, we might want to look at the derivatives of curves, or the tangent spaces of a Lorentzian manifold. Can they be generalized if we choose another topology? In (pseudo-)Riemannian geometry we have a nice notion of an exponential map, a diffeomorphism from the tangent space to the manifold that preserves the lengths of geodesics through the base point. Again, if we use $\mathbb{R}^n$ as the tangent space, this can not be a diffeomorphism anymore if we change the topology of the manifold. Can we choose another notion of a tangent space such that the exponential map is a local diffeomorphism everywhere? The answer again turns out to be no, but if we only demand that the exponential map is a homeomorphism on the *geodesics*, then it might be possible to define such a tangent space.
At any rate, the tangent space is also in particular a vector space, and independently of whether we can make the exponential map diffeomorphic, it might be interesting to see whether we can adept the vector space structure. (This might be related to the open question at the end of exercise 3 of the course.) In this way we might be able to at least define derivatives of geodesics. But there are more reasons why we might want to change the vector space structure. For example, we might want to incorporate the physical notion of addition of velocities in Minkowski space into the vector space addition structure.

In this essay we first take a look at the different topologies that have been considered in the past by Zeeman [4], Nanda [5, 6, 7], Göbel [2] and Hawking, King and McCarthy [8], and we note their advantages and disadvantages. We shall see how these results contradict the construction of a causal manifold. We also give a possible answer to the question of why no metrizable causal topologies have been found and we present a few new examples of topologies that we find might be interesting to study. After this, we take a closer look at defining a different vector space structure on the tangent spaces. We see whether we can replace the vector space structure by an augmentation of the structure of a gyrospace that was introduced by Ungar [9], that incorporates the modified notion of addition of velocities in special and general relativity. Then finally we discuss possible implications of these mathematical structures for the field of quantum gravity.

"Causal" Topologies

In this section we give a summary of some of the properties of a few of the topologies that have been considered in the past as causal topologies on spacetime, in near chronological order. It is evident that this list should start with Zeeman, as he was the first to publish on the matter of the topology of Minkowski space and also the first to express causality as a mathematical structure of partial ordering. But let us be clear that what follows is not a list of *all* the topologies that have been proposed afterwards, just of those that have been discussed most in literature.

The fine topology
Sir Christopher Zeeman was the first to give a possible solution to the problem of a causal topology on Minkowski space. In 1966 he proposed the *fine topology* on four-dimensional Minkowski space M, which is defined to be the finest topology (with the most open sets) such that the restriction of the topology to every spacelike hyperplane is the 3-dimensional Euclidean topology and to every time axis is the 1-dimensional Euclidean topology [4]. He proved that the homeomorphism group of this space is the *causality group* and that therefore the light cone at each point can be deduced from the topology alone.
The causality group was defined by Zeeman in [10] to be the group of homeomorphisms of Minkowski space with the Euclidean topology that leave the partial ordering of causality invariant. He proved that the causality group equals the group generated by orthochronous Lorentz transformations (rotations, boosts and reflections in space), translations and dilatations (multiplications by a scalar). Note that this topology is intrinsically linked to the Euclidean topology because the causality group is a subgroup of the homeomorphism group of $\mathbb{R}^n$.
A timelike path through this space is continuous if and only if it is piecewise linear, because then every piece is part of a time axis that is by definition homeomorphic to the piece of the interval from which the curve is defined in $\mathbb{R}$. A lightlike path, however, is never continuous, since the restriction topology on the lightlike surfaces is the discrete topology and therefore an interval in $\mathbb{R}$ can never be homeomorphic to it. This intuitively agrees with the fact that we have no evidence of a photon other than the events of its emission and absorption, according to Zeeman.
There are some downsides to this topology, however. Although it is Hausdorff, connected and locally connected, it is not normal, locally compact or first countable: it does not have a countable neighborhood basis. This is well demonstrated in the fact that, precisely because the discrete topology is induced on a lightlike surface, such a surface has zero Lebesgue covering dimension[2], with every point disconnected from the other points. This might be "physical", since it implies there is no topological information to be found on light rays, but it makes the fine topology very hard to work with.
A recent explicit study of all the properties the fine topology can be found in an article by Dossena [11]. Dossena also proves in this work that the 2-dimensional Minkowski space with the fine topology is not simply connected: not all paths are not continuously contractable. This result can probably be generalized to higher dimensions. In what measure a space is simply connected has a lot of topological consequences, for example in the study of the fundamental group. Furthermore, whether a path is contractible is probably also relevant to certain area's of quantum field theory. It is unclear at this point whether or not it is physically attractive to have a simply connected space.

[2]The Lebesgue covering dimension is the smallest number n such that every open covering can be refined to a cover where every point is contained in at most $n + 1$ sets.

The t-topology

Zeeman also proposed a few alternative topologies in his 1966 article [4]. The first two are finer than the fine topology. The t-topology is the finest topology such that the Euclidean topology is induced on the time axes only. The spacelike planes now get a discrete topology as well as the lightlike surfaces. It still holds that the only continuous timelike paths are piecewise linear. Any continuous path that is not necessarily directed positively in time at every point, is always a piecewise linear path that can switch between being timelike or anti-timelike, exactly like the Feynman path of, for example, an electron. Note that these paths are also continuous in the fine topology, but in the t-topology they are the only continuous paths.

Nanda showed in 1970 that the homeomorphism group of Minkowski space with the t-topology is again the causality group [5]. Recently the t-topology was revisited by Agrawal and Shrivastava in [12], and they proved that 2-dimensional Minkowski space with the t-topology is not simply connected. This result can, again, probably be generalized to higher dimensions.

The s-topology

Zeeman defined the s-topology to be the finest topology on M such that the Euclidean topology is induced only on the spacial hyperplanes. The time axes will have the discrete topology just like the lightlike surfaces. The space is Hausdorff, but not normal, not locally compact and not second countable. Nanda showed that its homeomorphism group is again the causality group [5]. Domiaty showed in 1984 that the extension of the s-topology to arbitrary manifolds - as the finest topology such that any spacelike hypersurface has the Euclidean topology - is 0-semimetrizable, which is a rather nice property for a space that is not metrizable [13].

The proof of Agrawal and Shrivastava that the 2-dimensional t-topology is not simply connected can probably be extended to the 2-dimensional s-topology [12].

The order topology

Another, very different, topology on Minkowski space that has the causality group as a homeomorphism group, is the order topology. It was introduced in the PhD thesis of Vroegindeweij [14], and Nanda and Panda proved that it is, unlike the above examples, simply connected [6]. The order topology is generated by the sets that are the complete forward light cone of a point, excluding the lightlike surfaces and including the point itself. It is so coarse that it does not contain the Euclidean topology and it is not Hausdorff. Note, however, that the induced topology on spacelike hypersurfaces is again the discrete topology, as the intersection of the forward light cone with a hyperspace through that point is the point itself.

Göbels extension

In 1976, Göbel extended the fine topology of Zeeman to an arbitrary Lorentzian manifold. He defined the *Zeeman topology* to be the finest topology on a Lorentzian manifold such that every restriction to a spacelike hypersurface or a timelike geodesic is locally Euclidean [2]. For the moment, we will refer to such a space a *Zeeman manifold.*

Göbel showed that the group of homeomorphisms of such a spacetime is the group of homothetic transformations, or constant conformal transformations. This means that two spaces are homeomorphic if and only if they are isometric up to a factor, because if a homothety between the spaces exists, we can always compose it with an another homothety from one of the spaces to itself that cancels the factor and the resulting composition will be an isometry. This also means that the pseudo-metric can be reconstructed up to a constant from the Zeeman topology alone.

Proposition: A Zeeman manifold is only locally homeomorphic to $\mathbb{M}^n$ with the fine topology if it is flat.

Proof: Suppose M is locally homeomorphic to $\mathbb{M}^n$. We restrict ourselves to one patch $U \subset M$ that is homeomorphic to $V \subset \mathbb{M}^n$. These are both spaces endowed with a Zeeman topology, and they are homeomorphic, therefore they are isometric up to a constant. Since $\mathbb{M}^n$ and therefore V have a flat metric, so does U. This holds for every homeomorphic patch, so M is flat everywhere.
Q.E.D.

This result is a direct consequence of the fact that the topology determines so much of the geometry of the space. It also implies that if we would try to give a curved Zeeman manifold a tangent space at each point, neither $\mathbb{R}^n$, nor $\mathbb{M}^n$ with the Zeeman topology, nor any other space will give us a diffeomorphic exponential map at every point: the topological structure of the space itself is not locally the same at every point because it depends on the geometry.

The path topology
Hawking, King and McCarthy were not satisfied with the fact that only piecewise straight lines or piecewise geodesics are continuous in the fine and Zeeman topologies. They argued that even in a flat spacetime, particles might be experiencing forces of another nature than gravity, and therefore follow an accelerated path. So they defined the *path topology* to be the finest topology such that any "continuous" timelike path is continuous [8]. Here the first notion of continuous is in the sense of the Euclidean topology and the second in the sense of the path topology. The homeomorphism group of the path topology is not the causality group but the larger *conformal group* of transformations that induce a conformal transformation on the pseudo-metric, which Hawking et al. found more physically appealing.
The path topology is defined in flat spacetime as well as in curved spacetime. Agrawal and Shrivastava write in their article [12] that a referee had noted that on 4-dimensional Minkowski space, the path topology is the same as the t-topology. They pose this claim without further proof. We suspect however that this claim is in fact false, since Zeeman proves in his article [4] that a finite continuous timelike path in the fine topology is always a connected sequence of a *finite* number of straight lines, and it seems like this result can be generalized to the t-topology. This would mean that an accelerating path is not continuous, while in the path topology it is - by definition. Zeeman actually claims that he has generalized the result in his article, but does not give the proof. However, for the fine topology the proof makes use of Zeno sequences - sequences that converge in the Euclidean but not in the fine topology - and these can also be used in the case of the t-topology.

In 2010 Low proved that n-dimensional Minkowski space with the path topology is not simply connected [15]. In fact, he proves that the space is "as non-simply connected as it gets", showing that no two closed Feynman paths are homotopic. A nice feature of the path topology, however, is that it is first countable: it has a countable basis of neighborhoods. These "balls" around a point x can be constructed by taking the intersection of a normal Euclidean ball with radius r with the open forward and backward light cone of x and adding x itself. Taking the collection of radii to be $\{\frac{1}{n} : n \in \mathbb{N}\}$, we get a countable basis. Note that the restriction of such a ball to a spacelike surface through x is exactly the point x itself, so that we indeed get the discreet topology on spacelike surfaces.

The A-Topology
Nanda thought it was unnatural that in the fine, t- and s-topologies, light is deprived of the privilege of traveling on a continuous path. Therefore he defined the *A-topology*, a subset of the fine topology, as the finest topology such that the 1-dimensional Euclidean topology is induced on timelike *and* lightlike axes, while the 3-dimensional Euclidean topology is induced on the spacelike hyperplanes [7]. Since in this topology every straight line gets the 1-dimensional Euclidean topology, all straight lines will actually be 1-dimensional, unlike in the fine, t- and s-topologies. It is not clear whether this also holds for all 2- and 3-dimensional (hyper)surfaces.

Alexandrov topology
We can view a Lorentz manifold as a partially ordered set (poset) with causality as a partial order, and as such we can endow it with the poset or Alexandrov topology. This topology is generated by intersections of the open past and future light cones of the points in the space, so sets of (elongated) rhombi. In general it is coarser than the locally Euclidean manifold topology, but it is equal to the manifold topology if and only if a space is strongly causal. Strongly causal means that in the manifold topology, every point has arbitrarily small neighborhoods that are causally convex, meaning that light- or timelike geodesics pass through it only once [16]. Minkowski space is strongly causal, so the Alexandrov topology reduces to the Euclidean topology for Minkowski space, and then the definition of a causal manifold reduces to that of a normal manifold. Obviously we then loose the property that the topology is "casual": we can not recover the light cones from the topology. McWilliams showed in 1980 that the Alexandrov topology is also complete exactly when the spacetime is strongly causal [17].

Metrizability
None of the above topologies are metrizable, and in [13] Domiaty wonders why no metrizable topologies on Lorentzian manifolds have been found. But for a compact metric space holds that if it is finite-dimensional, then it can be embedded in $\mathbb{R}^{2n+1}$ with the Euclidean topology [18]. So if we want to construct a *compact* Lorentzian space with a new topology, this implies that the topology cannot be metrizable and finite-dimensional at the same time, or it will be equal to the Euclidean topology.

Causal manifolds?

Why it would be very nice to define a causal manifold
As has been explained in the introduction, it would be very nice to extend the concept of a manifold to causal spaces that have the causality group or some other "causal" group as the homeomorphism group. One of the advantages of such a construction would be that we can define a continuous coordinate system as a map from the manifold to some "Minkowski space" that is locally homeomorphic to the manifold. Another nice feature would be that we would have this Minkowski space around that could then also play the role of a tangent space, with an exponential map that is locally diffeomorphic. Having defined these structures, we might be able to do extend our knowledge of differential and Riemannian geometry, and define concepts like curvature, and make sure that the theorems from Riemannian geometry all hold nicely. In fact, it is not clear what the use is of defining a new topology on Lorentzian manifolds, if we can not do this. What kind of mathematics can we do on this spaces without being able to define continuous coordinates or tangent spaces?

Why the causal manifold doesn't work
Unfortunately, it is not possible to define such a causal manifold that is non trivial. The reason for this is that the causal structure itself already determines too much of the geometrical structure of the manifold. This is shown by the fact that, as Zeeman proved in [10], the only homeomorphisms that leave the causal structure intact are the elements of the causality group. But these transformations do not allow for the space to be homeomorphic to a space with a pseudo-metric that is not related to the original one by a causal transformation. Starting out with a flat space, the only spaces that will be (locally) homeomorphic are flat too.
Even if we extend the causality group to the conformal group like Hawking et al., the spaces can only be homeomorphic if they are related by a conformal transformation. A general curved spacetime will not be locally conformally the same in all small enough neighborhoods, and therefore it will not locally be homeomorphic to itself.
In the next chapter we see how we might still be able to define tangent spaces that are not locally homeomorphic to the space, but for which the exponential map restricted to the straight lines, which are mapped to geodesics through the base point, is a homeomorphism of curves.

Something that might still be interesting?
One thing that might still be interesting to study, is whether we can define a topology that is a little less strict than the previous ones, and maybe more like the manifold topology. We know that it has to be fairly restrictive in order to be "causal" - to have the causality group or the conformal group as a homeomorphism group. But maybe we can define a topology that is *the coarsest* topology such that a condition holds, instead of *the finest*. We give a few examples of topologies on $\mathbb{M}^n$ that we think might be interesting to study:

Example 1a: The coarsest topology containing the *Euclidean topology*, such that the homeomorphism group of the space is the conformal group. We are not sure whether it can be determined what this topology is, since it is not clear whether and how we can construct all the topologies that have a particular homeomorphism group [5]. The condition that the Euclidean topology is included ensures that the space is Hausdorff, which we know we should restrict to explicitly because there is at least one causal topology that is not Hausdorff: the order topology.

Example 1b: The coarsest *Hausdorff* topology such that the homeomorphism group of the space is the causality group. It is not clear why exactly the Euclidean topology should be a subset of a causal topology, other than in order to make it Hausdorff. It might turn out that in order to be Hausdorff, the topology *needs* to contain the Euclidean topology, in which case Example 1b equals Example 1a. If this is not the case, then it is still not clear whether it is possible to find this topology.

Example 2: The finest topology such that the homeomorphism group of the space is the causality group, and such there is some notion of a topological dimension that gives any subspace the same dimension as it has in the Euclidean case (*dimensionality condition*). The A-topology is possibly a topology that satisfies this dimensionality condition, but is then it is probably not the finest topology to do so.
It is quite probable that Example 1a satisfies the dimensionality condition, since adding more sets to the Euclidean topology can only lower the Lebesgue covering dimension. Since we add as little sets as possible in Example 1a, it is obvious that, if a causal topology can be constructed such that the dimensionality condition for the Lebesgue dimension is satisfied, Example 1a will satisfy it.
Note that for spaces that are not metrizable, there are a number of different notions of dimension in topology. In fact, there is a whole field devoted to the study of dimensional invariants in topology: Dimension Theory.

Example 3: A (the?) n-dimensional Hausdorff topology such that a certain “causal” *vector space structure* is continuous. This is in analogy with $\mathbb{R}^n$, where the topology of the topological vector space, when asked to be Hausdorff and finite-dimensional, is exactly the Euclidean topology. In the next section we will study what is meant here by a causal vector space structure.

The Minkowski vector space

A tangent space in a point on a manifold can be viewed as the space of equivalence classes - derivatives - of nice curves through that point, with a certain addition structure: addition of velocities. If we change the topology of the manifold, we change what curves are to be considered as nice curves.

“Nice” curves
In a space with a Euclidean topology, a curve is continuous precisely when it is connected. This is because we define a curve as the image of an (injective[3])

[3]If the map is not injective, then we can change the dimension of the image. A constant curve or a plane-filling curve for example is not injective. We use the theorem that a continuous bijection from a compact to a Hausdorff space is a homeomorphism.

map from a Euclidean interval into the space. The restriction topology on a 1-dimensional subspace in a Euclidean space is Euclidean exactly when the subspace is connected, so then the curve can be the image of a homeomorphic map. So the notion of a "nice" curve in a manifold is not very restrictive.
When we give the Minkowski space a different topology, then curves might not be continuous while they are connected, because the restricted topology is not Euclidean. This mathematical artefact is used to define a few of the topologies that were mentioned before. But it can happen that the restriction topology on the curve is not the Euclidean topology, while the curve can in some sense be considered to be "nice". Here I mean by nice that they are, for example, homeomorphic to a timelike, spacelike or lightlike line in $\mathbb{M}^n$, and they indeed have that same character as a curve in the manifold.
We might be able to use "intervals" in Minkowski space to parametrize curves instead of intervals of $\mathbb{R}$, and then these curves *can* be considered to be continuous. We then do not use homeomorphisms of open coordinate patches, which as we have shown are not available, but just the homeomorphisms of topologies that are induced on the one-dimensional "curves" and "intervals".

Tangent space
A restriction we might now be able to impose on defining a causal topology on a Lorentzian manifold, is that it agrees with the causal topology of $\mathbb{M}^n$ in the following way: at every point, the geodesics in every direction are homeomorphic with straight lines in $\mathbb{M}^n$. This resembles the definition of a causal manifold, but whereas that definition was not possible because the space is not locally isometric, we know that we do have isometries of geodesics to straight lines in a tangent space at every point. This is what the exponential map tells us: at every point, we can project a straight line in the tangent space to the manifold. This locally gives us a geodesic through the point with the same length as the straight line at every point:

$$\exp:\ tX \mapsto \gamma_X(t),$$

where X is a vector in the tangent space that is also the derivative of the geodesic γ_X at 0, and t is the parameter along both curves.
We have already seen that we cannot construct a tangent space at each point such that the exponential map is a homeomorphism on an open neighborhood. But the one-dimensional lines do not have any curvature. In a one-dimensional space we can only vary distances conformally. Therefore it might be possible to construct a tangent space that is the same at each point such that the projection of straight lines into geodesics is at least a homeomorphism.

Constructing a tangent bundle
To construct a tangent bundle, which is a space of derivatives of nice curves at every point, we should first look at the simplest example of a causal space: $\mathbb{M}^n$ itself. When trying to draw the analogy with $\mathbb{R}^n$, we note that the tangent space of $\mathbb{R}^n$ at each point is $\mathbb{R}^n$ itself. Extending this idea to $\mathbb{M}^n$ is a logical step. But the tangent space should be a *vector space*, in which each vector represents the infinitesimal change of a path on the manifold through the base point of the tangent space. So what if we make $\mathbb{M}^n$ a vector space by defining vector addition and scalar multiplication to be the same thing as in $\mathbb{R}^n$? This is what

is usually done, for example in [4]. We argue here that it might be possible to change this structure.

$\mathbb{R}^n$ can be viewed as a vector space and a manifold at the same time, because the corresponding topologies coincide. On the one hand, $\mathbb{R}^n$ can be endowed with the Euclidean topology induced by the Euclidean norm, which coincides with the topology of any smooth deformation or manifold covering of $\mathbb{R}^n$ and is therefore not really bound to the Euclidean metric itself per se. On the other hand, we have the abstract concept of a topological vector space: a vector space that comes with a topology which makes vector addition and scalar multiplication continuous. Demanding that the topology of such a topological vector space is finite-dimensional and Hausdorff, automatically makes it homeomorphic to $\mathbb{R}^n$ endowed with the Euclidean topology, where n is the dimension of the vector space. So in the case of $\mathbb{R}^n$, the concepts of manifold structure and vector space structure really coincide at the deepest level. This means for Riemannian manifolds that the exponential map, that gives a local diffeomorphism between a piece of the tangent space and a piece of the manifold, can also carry the vector space structure down to the base point on the manifold, enabling us to infinitesimally "add" geodesics (add velocities) at the base point.

We already know that a part of this structure can not be generalized to Lorentz manifolds with a causal topology, because they are not locally homeomorphic to some tangent space that is the same everywhere. This is because the topology will depend locally on the local geometry of the space. It might be possible to extend the structure to Minkowski space itself, however.
We now give three arguments why we should change the vector space structure on Minkowski space if we change the topology:

1. As a topological vector space, the topology that is induced on Minkowski space with the Euclidean vector space structure is the Euclidean topology. Therefore we again have a discordance in topologies.

2. Addition of certain vectors in the tangent space can be viewed as addition of velocities. We know from special relativity that addition of velocities in Minkowski space differs from the usual addition of vectors.

3. With the normal addition we can add a spacelike and a timelike vector to obtain a timelike, lightlike or a spacelike vector. This does not seem very physical.

We could therefore try to replace the $\mathbb{R}^n$ vector space structure on $\mathbb{M}^n$ with a "causal" vector space structure that reduces to "Minkowskian" addition on the proper velocities, which are positively oriented timelike vectors with negative norm 1.

Adding proper velocities in 2D
Let us first look at addition of velocities in two dimensions. In two dimensional Minkowski space, the rule for adding two velocities, viewed from a third frame, is:

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2},$$

where we have set the velocity of light equal to 1. This formula comes from applying a Lorentz boost with velocity v_1 to the spacetime proper velocity that belongs to v_2 or vice versa. The proper velocity vector that belongs to a velocity v_u is:

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \gamma_u \\ \gamma_u v_u \end{pmatrix}.$$

We can therefore write the addition law of proper velocities as:

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \oplus_C \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \Lambda(v_u)w = \begin{pmatrix} \gamma_u & \gamma_u v_u \\ \gamma_u v_u & \gamma_u \end{pmatrix} \begin{pmatrix} \gamma_w \\ \gamma_w v_w \end{pmatrix} = \begin{pmatrix} u_0 & u_1 \\ u_1 & u_0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix},$$

where $\oplus_C$ stands for "causal"[4] addition in a Minkowski space and $\gamma_i = \frac{1}{\sqrt{1-v_i^2}}$.

More dimensions
Unfortunately, in more than two dimensions, this addition process is not associative and not even commutative. What velocity vector we end up with, when we add two velocities, depends on which frame we boost to first. This effect can been seen as a relativistic correction to the spin of an elementary particle or to the rotation of a relativistic gyroscope. It is called *Thomas precession* and was discovered by Thomas in 1926 [19]. In mathematical terms, it stems from the fact that the composition of two non-parallel Lorentz boosts is not a pure boost, but a combination of a boost and a rotation, called the Thomas rotation. The formula for adding two velocities in more than two dimensions becomes:

$$\mathbf{v}_1 \oplus_C \mathbf{v}_2 = \frac{1}{1 + \mathbf{v}_1 \cdot \mathbf{v}_2} \left(\mathbf{v}_1 + \frac{\mathbf{v}_2}{\gamma_1} + \frac{\gamma_1}{1+\gamma_1} (\mathbf{v}_1 \cdot \mathbf{v}_2) \mathbf{v}_1 \right).$$

While this operation is not associative or commutative, it does obey the *gyrocommutative* and left and right *gyroassociative* laws [9]:

$$\begin{aligned} \mathbf{v}_1 \oplus_C \mathbf{v}_2 &= \mathrm{gyr}[\mathbf{v}_1; \mathbf{v}_2](\mathbf{v}_2 \oplus_C \mathbf{v}_1) \\ \mathbf{v}_1 \oplus_C (\mathbf{v}_2 \oplus_C \mathbf{v}_3) &= (\mathbf{v}_1 \oplus_C \mathbf{v}_2) \oplus_C \mathrm{gyr}[\mathbf{v}_1; \mathbf{v}_2]\mathbf{v}_3 \\ (\mathbf{v}_1 \oplus_C \mathbf{v}_2) \oplus_C \mathbf{v}_3 &= \mathrm{gyr}[\mathbf{v}_1; \mathbf{v}_2]\mathbf{v}_1 \oplus_C (\mathbf{v}_2 \oplus_C \mathbf{v}_3), \end{aligned}$$

where $\mathrm{gyr}[\mathbf{u}; \mathbf{v}]$ is the Thomas rotation that takes $\mathbf{v} \oplus_C \mathbf{u}$ to $\mathbf{u} \oplus_C \mathbf{v}$. Together the set of proper velocities with relativistic addition therefore form a *gyrogroup*.

Scalar multiplication
In a gyrogroup, we can also define a notion of scalar multiplication, for which holds:

$$n \odot_C \mathbf{v} = \underbrace{\mathbf{v} \oplus_C \ldots \oplus_C \mathbf{v}}_{n \text{ times}}.$$

It turns out that in this case the formula for this scalar multiplication is [9]:

$$\begin{aligned} r \odot_C \mathbf{v} &= \tanh(r \operatorname{arctanh}(\|\mathbf{v}\|)) \\ &= \frac{(1+\|\mathbf{v}\|)^r - (1-\|\mathbf{v}\|)^r}{(1+\|\mathbf{v}\|)^r + (1-\|\mathbf{v}\|)^r} \frac{\mathbf{v}}{\|\mathbf{v}\|}. \end{aligned}$$

[4]Ungar denotes this product with a subscript E for "Einstein". This would be utterly confusing in our case, as we want to emphasize the contrast with "Euclidean". On the other hand, the subscript M for "Minkowski" is used by Ungar to denote "Möbius".

Together with the addition and scalar multiplication structure, we can call the space a *gyrospace*, as a generalization of a vector space.

Proper vectors along geodesics
We now have a gyrospace of proper velocities that are exactly the derivatives of positively oriented timelike geodesics in Minkowski space, along with their Minkowskian addition structure. But we want to describe the whole tangent space, so the derivatives along any straight line through the origin in Minkowski space. First we will look at the analogy of a proper velocity along paths that are not timelike and positively oriented. Let us call any vector with absolute norm 1 a proper vector. We get the set of negatively oriented timelike vectors and all the spacelike vectors with norm one.
The negatively oriented timelike vectors can be defined to transform in the same way as the positively oriented ones. Now consider the spacial proper vectors. We know that when we make a Lorentz boost to a certain frame, the spacial coordinate in the direction of the boost transforms in a similar but opposite way compared to the timelike vector. So the spacial directions can probably also be described by something that looks like a gyrospace.

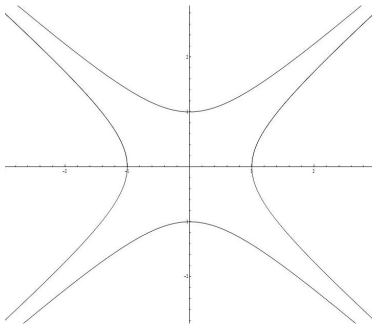

Figure 1: The collection of proper vectors in $\mathbb{M}^2$

Note that in more than two dimensions, these proper vectors form three separate components in $\mathbb{M}^n$: the past and future light cones and the "space cone". The lightlike vectors are not in the set of proper vectors since they have norm zero. But we can probably define a similar structure for lightlike vectors. We think it would be a nice to treat all these different pieces separately when forming a complete "vector space", and leave the addition of vectors with a different character undefined.

Extending to the whole of $\mathbb{M}^n$
We have now only considered the vectors of absolute norm 1. To extend the structure to the whole of $\mathbb{M}^n$, we might be able to just tensor these vectors with the interval $[0, \infty)$. If we still want to give $\mathbb{M}^n$ another topology, then we can give $[0, \infty)$ the appropriate topology, possibly a different one for the timelike, spacelike and lightlike parts. In this way we cover the whole of $\mathbb{M}^n$, possibly in such a way that we can use it as a tangent space (that maps homeomorphically to geodesics) at points of a Lorentzian manifold, either with a new causal topology or with the old manifold topology.

Possible implications for Quantum Gravity

Diffeomorphism group
A nice feature of redefining the topology on spacetimes would be that we then might be able to use the homeomorphism (or diffeomorphism) group of the space in the construction of a theory of quantum gravity [3].

Wick rotation
If we choose a different topology for a Lorentz manifold, then this implies that Wick rotation to a normal Riemannian manifold, with the locally Euclidean topology, is not a continuous transformation. In the process of Wick rotation we replace our time-coordinate by an imaginary variable, thereby transforming the pseudo-metric into a Riemannian metric. We could of course give the Wick rotated space the same causal topology, but then this topology would not agree with the metric topology, and the space would not be a Riemannian manifold either. At any rate, if we endow the Lorentz manifold with another topology, then it is not at any level isomorphic to a Riemannian manifold anymore. But this does not necessarily imply that the analogy between a causal manifold and its Riemannian counterpart does not have any meaning at all.

Triangulations
A triangulation of a topological space can be viewed as a simplicial cell decomposition of the space. If we endow a spacetime with another topology, then it will probably not be possible to make a cellular decomposition where the top-dimension cells are homeomorphic with balls in $\mathbb{R}^{\mathrm{n}}$. In order to make a cell decomposition of the space, we would then probably need to incorporate causality in (at least) the top cells. If we then ask that such a top cell is a simplex, then we can look at the properties of the faces and lower dimensional boundaries: these might be spacelike, the one-dimensional edges might be lightlike or timelike, or they might change in character. This structure resembles the procedure in causal dynamical triangulations, where we demand that some of the faces of simplices are explicitly spacelike [20].

Linearity
If we change the vector space structure on Minkowski space, then the definition of the word *linear* will change. Any theory that is linear, might have to be rewritten into a theory that is "causal linear". Note that this notion of linearity, along with the notion of addition of velocities, probably reduces to normal linearity and velocity addition in the non-relativistic limit.

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